Macroeconomics 1 (2/7)

The growth model with an endogenous saving rate (Cass-Koopmans-Ramsey)

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Cass, Koopmans, Ramsey

- "**Cass-Koopmans-Ramsey model**" $\equiv$ model derived from the contributions of Ramsey (1928), Cass (1965) and Koopmans (1965).
- **Franck P. Ramsey**: English mathematician, logician and economist, born in Cambridge in 1903, deceased in London in 1930.
- **David Cass**: Americain economist, born in 1937 in Honolulu, deceased in 2008 in Philadelphia, professor the University of Pennsylvania since 1974.
- **Tjalling C. Koopmans**: Dutch and American economist, born in 1910 in 's-Graveland, deceased in 1985 in New Haven, professor at Yale University since 1955, co-laureate (with Leonid V. Kantorovich) of the Sveriges Riksbank's prize in economic sciences in memory of Alfred Nobel in 1975 "*for their contributions to the theory of optimum allocation of resources*".

Motivation

- The Cass-Koopmans-Ramsey model endogenizes the saving rate of the Solow-Swan model.

- This endogenization leads to a more refined positive and normative analysis, in particular of the effect of economic policies.

- Does this change the main predictions of the Solow-Swan model about
  - the determinants of long-term growth,
  - conditional convergence,
  - the possibility of dynamic inefficiency?

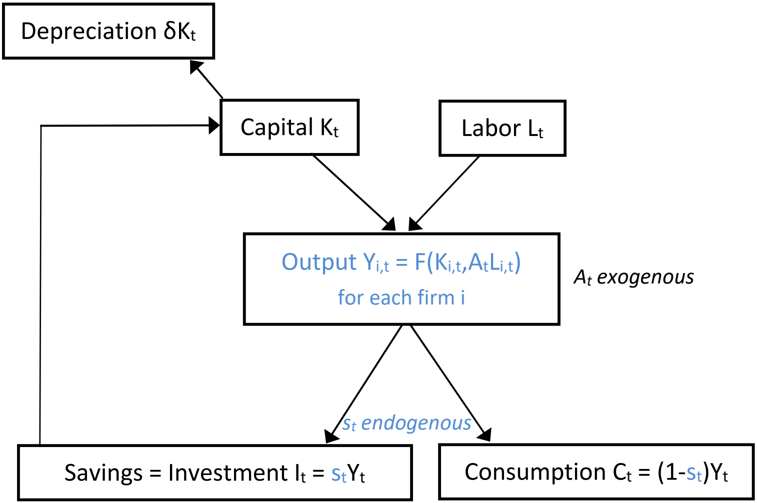
- No, no, and yes.

General overview of the model I *

- **Firms** rent capital and employ labor to produce goods.
- **Households** own capital and supply labor.
- The goods produced by firms are used for households' consumption and investment in new capital.
- The **saving rate** is **endogenous**, optimally chosen by **households**.
- Capital evolves over time due to investment and capital depreciation.

(In the pages whose title is followed by an asterisk,
in blue: changes from Chapter 1.)

General overview of the model II *



Good, private agents, markets

- **Only one type of good**, used for
  - consumption,
  - investment.

- **Two types of private agents**:
  - households,
  - firms.

- **Four markets**:
  - goods markets,
  - labor market,
  - capital market,
  - loans market.

Origin of supply and demand on each market

- **Goods market**:
  - *supply* from firms,
  - *demand* from households.
- **Labor market**:
  - *supply* from households,
  - *demand* from firms.
- **Capital market**:
  - *supply* from households,
  - *demand* from firms.
- **Loans market**:
  - *supply* from households,
  - *demand* from households.

Perfectly competitive markets

- These four markets are assumed to be **perfectly competitive**, that is to say that they satisfy the following five conditions:
  1. market atomicity,
  2. product homogeneity,
  3. market transparency,
  4. free entry and exit,
  5. free movement of inputs.
- The atomicity condition says that the supply or demand of each agent is negligible compared to total supply or demand, and implies that no single agent can influence prices.
- In Chapter 5, we will consider markets that are imperfectly competitive as they do not satisfy the atomicity condition.

Exogenous variables *

- **Neither flows nor stocks**:
  - continuous time, indexed by t,
  - price of goods $\equiv$ numéraire $= 1$,
  - (large) number of firms I,
  - ~~saving rate s, with $0 < s < 1$.~~
- **Flow**:
  - labor supply $= 1$ per person.
- **Stocks**:
  - aggregate initial capital $K_0 > 0$,
  - population $L_t = L_0 e^{nt}$, where $L_0 > 0$ and $n \geq 0$,
  - productivity parameter $A_t = A_0 e^{gt}$, where $A_0 > 0$ and $g \geq 0$.

Endogenous variables *

- **Prices**:
  - real usage cost of capital z_t,
  - real wage w_t,
  - real interest rate r_t.
- **Quantities – flows**:
  - output $Y_{i,t}$ of firm i,
  - labor demand $N_{i,t}$ of firm i,
  - aggregate output $Y_t \equiv \sum_{i=1}^{I} Y_{i,t}$,
  - aggregate labor demand $N_t \equiv \sum_{i=1}^{I} N_{i,t}$,
  - aggregate consumption C_t.
- **Quantities – stocks**:
  - capital $K_{i,t}$ of firm i (except at $t = 0$),
  - aggregate capital $K_t \equiv \sum_{i=1}^{I} K_{i,t}$ (except at $t = 0$),
  - real aggregate amount of assets B_t.

Partial and general equilibria

- **Partial equilibrium** $\equiv$ situation in which supply equals demand on a single market.
- **General equilibrium** $\equiv$ situation in which supply equals demand on all markets.

General-equilibrium conditions

- Each private agent solves their optimization problem: as all markets are perfectly competitive,
  - at each time $t \geq 0$, each firm i chooses $(Y_{i,t}, K_{i,t}, N_{i,t})$, as a function of the prices (w_t, z_t) that they consider as given, in order to maximize their *instantaneous* profit subject to constraints,
  - at time 0, the representative household chooses $(\frac{C_t}{L_t}, \frac{B_t}{L_t})_{t\geq 0}$, as a function of the prices $(w_t, z_t, r_t)_{t\geq 0}$ that they consider as given, in order to maximize their *intertemporal* utility (under perfect expectations) subject to constraints.

- Prices are such that each market is cleared at each time $t \geq 0$:
  - w_t clears the labor market: $N_t = L_t$,
  - z_t clears the capital market,
  - r_t clears the loan market.

Reminder about the actualized value of a future variable

- We consider a flow or a stock of goods, denoted by x_t for any time t, and two times t_1 and t_2 such that $t_1 < t_2$.
- The **actualized value** at time t_1 of x_{t_2} is defined as the value that x_t should take at time t_1 if x_t were placed on the loans market from time t_1 to time t_2 and took the value x_{t_2} at time t_2.
- In case x_{t_2} is an income, it is therefore the amount that can be borrowed and consumed at t_1 if the borrowing is reimbursed at t_2 with this income.
- If x_t were placed on the loans market, then its instantaneous growth rate at time t would be the real interest rate, denoted by r_t:

$$\frac{\dot{x}_t}{x_t} \equiv \lim_{dt \to 0^+} \frac{x_{t+dt} - x_t}{x_t dt} = r_t.$$

- Integrating from t_1 to t_2, we get $x_{t_2} = x_{t_1} e^{\int_{t_1}^{t_2} r_t dt}$.
- The actualized value at time t_1 of x_{t_2} is therefore $x_{t_2} e^{-\int_{t_1}^{t_2} r_t dt}$.

Chapter outline

Equilibrium conditions

Firms' optimization problem

- Output of each firm i: $Y_{i,t} = F(K_{i,t}, A_t N_{i,t})$, where the production function F has the same properties as in Chapter 1.
- Each firm rents at each time the capital stock that they want to use, and there is no capital adjustment cost (in particular investment is reversible).
- So, at each time t, firm i chooses $K_{i,t}$ and $N_{i,t}$ in order to maximize their **instantaneous** profit

$$F(K_{i,t}, A_t N_{i,t}) - z_t K_{i,t} - w_t N_{i,t},$$

considering z_t and w_t as given.

First-order conditions

- Denoting by F_j the partial derivative of F with respect to its j^{th} argument for $j \in \{1,2\}$, we get the first-order conditions

$$\begin{aligned} F_1(K_{i,t}, A_t N_{i,t}) = z_t & \quad \text{(marginal productivity of capital = usage cost)}, \\ A_t F_2(K_{i,t}, A_t N_{i,t}) = w_t & \quad \text{(marginal productivity of labor = wage)}. \end{aligned}$$

- Using these conditions and Euler's theorem applied to Function F homogeneous of degree one, we get that the instantaneous profit is zero for any $K_{i,t}$ and $N_{i,t}$:

$$\begin{gathered} F(K_{i,t}, A_t N_{i,t}) - K_{i,t} F_1(K_{i,t}, A_t N_{i,t}) - A_t N_{i,t} F_2(K_{i,t}, A_t N_{i,t}) = 0 \\ \Longrightarrow F(K_{i,t}, A_t N_{i,t}) - z_t K_{i,t} - w_t N_{i,t} = 0. \end{gathered}$$

- **Leonhard P. Euler**: Swiss mathematician and physicist, born in 1707 in Basel, deceased in 1783 in Saint-Petersburg.

Households' intertemporal utility

- **Infinitely-lived representative household** or else dynastic lineage whose generations are linked together by bequest and altruism (this assumption will be relaxed in Chapter 7).

- At time 0, their **intertemporal utility** is

$$U_0 \equiv \int_0^{+\infty} e^{-\rho t} \frac{L_t}{L_0} u(c_t) dt = \int_0^{+\infty} e^{-(\rho - n)t} u(c_t) dt$$

where

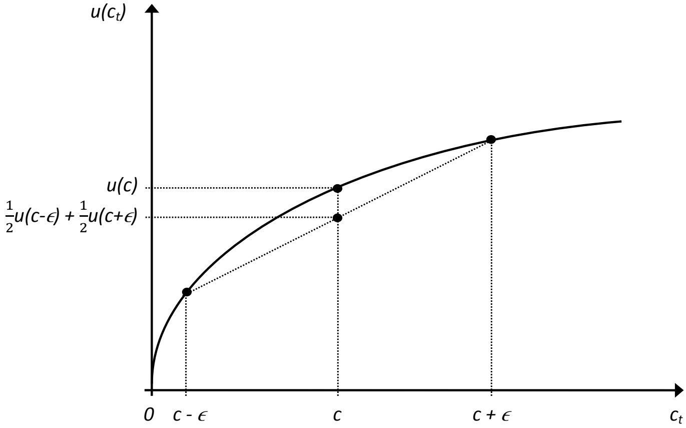
  - ρ is the rate of time preference ($\rho > 0$),
  - $c_t \equiv \frac{C_t}{L_t}$ is per-capita consumption,
  - u is the instantaneous-utility function,

if the (large) constant number of households is normalized to L_0 (without any loss in generality).

Properties of the instantaneous-utility function

1. u: $\mathbb{R}^+ \to \mathbb{R}$ (u can take negative values as it takes ordinal $-$ not cardinal $-$ values),
2. u is strictly **increasing**: $\forall x \in \mathbb{R}^+$, $u'(x) > 0$,
3. u is strictly **concave** (which implies a preference for smoothing consumption over time): $\forall x \in \mathbb{R}^+$, $u''(x) < 0$,
4. u satisfies the **Inada conditions**: $\lim_{x \to 0} u'(x) = +\infty$ and $\lim_{x \to +\infty} u'(x) = 0$.

Strict concavity of u and consumption smoothing



Case 1: $c_t = c$. Case 2: $Proba[c_t = c - \epsilon] = Proba[c_t = c + \epsilon] = \frac{1}{2}$ with $0 < \epsilon < c$.

We have $u(c) > \frac{1}{2}u(c - \epsilon) + \frac{1}{2}u(c + \epsilon)$.

Households' assets

- Each household can hold two types of assets:
  - loans to other households (zero in equilibrium),
  - capital ownership titles.

- In equilibrium, households must be indifferent between these two asset types, so

$$\begin{aligned} r_t &\equiv \text{real interest rate on loans to households} \\ &= \text{real rate of return on ownership titles} \end{aligned}$$

(if it were a strict inequality, then all households would like to lend and none to borrow, or vice-versa).

- $B_t \equiv$ total amount of assets in units of goods.

- $b_t \equiv \frac{B_t}{L_t}$ total amount of assets in units of per-capita goods.

Households' budget constraint I

- As the representative household has access to the loans market, they can in some sense "intertemporally transfer" their wage incomes and thus choose a consumption path subject only to an **intertemporal budget constraint** of type

  actualized value at time 0 of future consumptions $\leq$ wealth at time 0 + actualized value at time 0 of future wage incomes.

- In aggregate terms, it is written as

$$\underbrace{\int_0^{+\infty} C_t e^{-\int_0^t r_\tau d\tau} dt}_{\substack{\text{what should be saved at time 0} \\ \text{to finance future consumption}}} \leq \underbrace{B_0}_{\substack{\text{wealth} \\ \text{at time 0}}} + \underbrace{\int_0^{+\infty} W_t e^{-\int_0^t r_\tau d\tau} dt}_{\substack{\text{what could be borrowed at time 0 and} \\ \text{reimbursed with future wage incomes}}} .$$

- It can be rewritten, in per-capita terms, as

$$\int_0^{+\infty} c_t e^{-\int_0^t (r_\tau - n) d\tau} dt \leq b_0 + \int_0^{+\infty} w_t e^{-\int_0^t (r_\tau - n) d\tau} dt.$$

Households' budget constraint II

- The **instantaneous budget constraint** of the representative household is, in aggregate terms,

$$\dot{B}_t = r_t B_t + w_t L_t - C_t.$$

- It can be rewritten, in per-capita terms, as

$$\dot{b}_t = (r_t - n) b_t + w_t - c_t.$$

- Re-arranging the terms and multiplying by $e^{-\int_0^t (r_\tau - n) d\tau}$, we get

$$\left[\dot{b}_t - (r_t - n)\, b_t\right] e^{-\int_0^t (r_\tau - n) d\tau} = (w_t - c_t)\, e^{-\int_0^t (r_\tau - n) d\tau}.$$

- Then, integrating from 0 to T,

$$b_T e^{-\int_0^T (r_\tau - n) d\tau} - b_0 = \int_0^T (w_t - c_t)\, e^{-\int_0^t (r_\tau - n) d\tau} dt.$$

Households' budget constraint III

- Going to the limit $T \to +\infty$ and re-arranging the terms, we get

$$\int_0^{+\infty} c_t e^{-\int_0^t (r_\tau - n) d\tau} dt = b_0 + \int_0^{+\infty} w_t e^{-\int_0^t (r_\tau - n) d\tau} dt - \lim_{T \to +\infty} \left[b_T e^{-\int_0^T (r_\tau - n) d\tau} \right].$$

- We thus get the **intertemporal budget constraint**

$$\int_0^{+\infty} c_t e^{-\int_0^t (r_\tau - n) d\tau} dt \leq b_0 + \int_0^{+\infty} w_t e^{-\int_0^t (r_\tau - n) d\tau} dt$$

$$\text{if and only if} \quad \lim_{T \to +\infty} \left[b_T e^{-\int_0^T (r_\tau - n) d\tau} \right] \geq 0$$

$$\text{or, equivalently, if and only if} \quad \lim_{T \to +\infty} \left(B_T e^{-\int_0^T r_\tau d\tau} \right) \geq 0.$$

Households' solvency constraint

- The condition
$$\lim_{T\to+\infty}\left(B_T e^{-\int_0^T r_\tau d\tau}\right)\geq 0$$
is households' **solvency constraint**.

- It imposes that the actualized value at time 0 of the total amount of assets in the long term must be non-negative.

- It implies that in the long term, total debt ($-B_T$ when $B_T<0$) cannot increase at a rate higher than or equal to the interest rate (r_T).

- It rules out the possibility of financial scheme in which each borrowing would be reimbursed with a new borrowing ("Ponzi scheme").

- **Carlo P.G.G.T. Ponzi**: American and Italian crook, born in 1882 in Lugo, deceased in 1949 in Rio de Janeiro.

Households' optimization problem

- For some given $(r_t, w_t)_{t\geq 0}$ and b_0,

$$\max_{(c_t)_{t\geq 0},(b_t)_{t>0}} \int_0^{+\infty} e^{-(\rho-n)t} u(c_t) dt$$

subject to the constraints

1. $\forall t \geq 0$, $c_t \geq 0$ (constraint of consumption non-negativity),
2. $\forall t \geq 0$, $\dot{b}_t = (r_t - n) b_t + w_t - c_t$ (instantaneous budget constraint),
3. $\lim_{t \to +\infty} \left[b_t e^{-\int_0^t (r_\tau - n) d\tau} \right] \geq 0$ (solvency constraint).

- This problem is difficult due to its **intertemporal** nature.

- Parable of Robinson Crusoe and barley grains.

General dynamic-optimization problem

- This problem is a particular case (among others that we will encounter throughout the course) of the following general problem: for a given $k_0 > 0$,

$$\max_{(c_{i,t})_{i\in\{1,...,m\},t\geq 0},(k_t)_{t>0}} \left\{ \int_0^{+\infty} U\left[(c_{i,t})_{i\in\{1,...,m\}}, k_t, t \right] dt \right\}$$

subject to the constraints

1. $\forall i \in \{1, ..., m\}$, $\forall t \geq 0$, $c_{i,t} \geq 0$,
2. $\forall t \geq 0$, $\dot{k}_t = G\left[(c_{i,t})_{i\in\{1,...,m\}}, k_t, t \right]$,
3. $\lim_{T\to+\infty} \left\{ k_T e^{-\int_0^T \frac{\partial G}{\partial k_t} dt} \right\} \geq 0$ ("households' case") or
3. $\forall t > 0$, $k_t \geq 0$ ("planner's case"),

where $m \in \mathbb{N}^*$, and where Functions U and G have certain properties.

Optimal-control theory I

- Optimal-control theory enables one to decompose this (general) intertemporal problem into **instantaneous** problems.

- We define the **Hamiltonian** associated to the problem by

$$H\left[(c_{i,t})_{i\in\{1,...,m\}}, k_t, \lambda_t, t\right] \equiv U\left[(c_{i,t})_{i\in\{1,...,m\}}, k_t, t\right] + \lambda_t G\left[(c_{i,t})_{i\in\{1,...,m\}}, k_t, t\right].$$

- **William R. Hamilton**: Irish mathematician, physicist and astronomer, born in 1805 in Dublin, deceased in 1865 in Dublin.

- We call
  - k_t the state variable (typically a stock),
  - $(c_{i,t})_{i\in\{1,...,m\}}$ the control variables (typically flows),
  - λ_t the costate variable.

Optimal-control theory II

- $\left[(c_{i,t})_{i\in\{1,...,m\},t\geq 0}, (k_t)_{t>0}\right]$ is a solution to the dynamic-optimization problem if and only if there exists $(\lambda_t)_{t\geq 0}$ such that

1. the constraints of the dynamic-optimization problem are satisfied,
2. $\forall t \geq 0$, $\lambda_t \geq 0$ (non-negativity of the costate variable),
3. $\forall j \in \{1, ..., m\}$, $\forall t \geq 0$, $\frac{\partial H}{\partial c_{j,t}} = 0$ (first-order conditions on the control variables),
4. $\forall t \geq 0$, $\frac{\partial H}{\partial k_t} = -\dot{\lambda}_t$ (costate equation),
5. $\lim\limits_{t\to+\infty} k_t \lambda_t = 0$ (transversality condition).

Applying optimal-control theory I

- We now apply optimal-control theory to households' optimization problem.
- We define the **Hamiltonian** associated to households' optimization problem by
$$H(c_t, b_t, \lambda_t, t) \equiv e^{-(\rho-n)t}u(c_t) + \lambda_t \left[(r_t - n)\, b_t + w_t - c_t\right],$$
where λ_t represents the utility at time 0 of one unit of savings at time t, so that $H(c_t, b_t, \lambda_t, t)$ represents the utility at time 0 of consumption and savings at time t.
- We call
  - b_t the state variable,
  - c_t the control variable,
  - λ_t the costate variable.

Applying optimal-control theory II

- $[(c_t)_{t\geq 0}, (b_t)_{t>0}]$ is a solution to households' optimization problem if and only if there exists $(\lambda_t)_{t\geq 0}$ such that

  1. $\forall t \geq 0$, $\frac{\partial H}{\partial c_t} = 0$ (first-order condition on the control variable),
  2. $\forall t \geq 0$, $\frac{\partial H}{\partial b_t} = -\dot{\lambda}_t$ (costate equation),
  3. $\forall t \geq 0$, $\dot{b}_t = (r_t - n)b_t + w_t - c_t$ (instantaneous budget constraint),
  4. $\lim_{t\to+\infty} b_t \lambda_t = 0$ (transversality condition),
  5. $\lim_{t\to+\infty} \left[b_t e^{-\int_0^t (r_\tau - n)d\tau}\right] \geq 0$ (solvency constraint),
  6. $\forall t \geq 0$, $c_t \geq 0$ and $\lambda_t \geq 0$ (non-negativity constraints).

- A brief interpretation of these optimality conditions is proposed in the appendix (for a more detailed interpretation, see the appendix of Chapter 1 of Aghion and Howitt, 1998).

Existence and uniqueness of the solution

- We show in three steps that there exists a unique solution:

1. use the first two conditions to get $(c_t)_{t>0}$ and $(\lambda_t)_{t\geq 0}$ as functions of c_0,
2. use the next two conditions to get $(b_t)_{t>0}$ and c_0,
3. show that the last two conditions are satisfied by the paths $(b_t)_{t>0}$, $(c_t)_{t\geq 0}$ and $(\lambda_t)_{t\geq 0}$ thus obtained.

Euler equation I

- The first condition implies $\lambda_t = e^{-(\rho-n)t}u'(c_t)$.

- Using the second condition, we then get

$$\frac{\dot{\overline{u'(c_t)}}}{u'(c_t)} = \rho - r_t$$

from which we deduce the **Euler equation** (or “Ramsey's optimal-savings rule”, or “Keynes-Ramsey rule”):

$$\frac{\dot{c}_t}{c_t} = \left[\frac{-u''(c_t)c_t}{u'(c_t)}\right]^{-1}(r_t - \rho).$$

- **John M. Keynes**: English economist, born in 1883 in Cambridge, deceased in 1946 in Firle.

Euler equation II

- There is a positive growth of per-capita consumption ($\frac{\dot{c}_t}{c_t} > 0$) if and only if the interest rate is higher than the rate of time preference ($r_t > \rho$).

- This growth is higher, the larger the **elasticity of intertemporal substitution** $\sigma(c_t)$, where

$$\sigma(c_t) \equiv \left[\frac{-u''(c_t)c_t}{u'(c_t)}\right]^{-1} > 0.$$

- The inverse of $\sigma(c_t)$ is equal to the opposite of the elasticity of marginal utility $u'(c_t)$ with respect to c_t,

$$\frac{1}{\sigma(c_t)} = \frac{-u''(c_t)c_t}{u'(c_t)} = \frac{-\frac{du'(c_t)}{u'(c_t)}}{\frac{dc_t}{c_t}},$$

  which measures the curvature of u at point c_t and is also called "**coefficient of relative risk aversion**" (for the reason mentioned on page 20).

Euler equation III

- For some given ρ and $\sigma(c_t)$, the higher the interest rate r_t, the more households invest, and therefore the higher the growth rate of c_t.

- For some given r_t and $\sigma(c_t)$, the higher the rate of time preference ρ, the less households invest, and therefore the lower the growth rate of c_t.

- For some given ρ and r_t, the higher the elasticity of intertemporal substitution $\sigma(c_t)$, the less households want to smooth c_t over time, and therefore the higher the growth rate of c_t.

Transversality condition

- Integrating the differential equation $\dot{\lambda}_t = -(r_t - n)\lambda_t$ (coming from the second condition), we get

$$\lambda_t = \lambda_0 e^{-\int_0^t (r_\tau - n) d\tau}$$

where $\lambda_0 = u'(c_0) > 0$ because c_0 is finite ($c_0 \leq \frac{Y_0}{L_0} < +\infty$).

- The transversality condition $\lim_{t \to +\infty} b_t \lambda_t = 0$ can therefore be rewritten as

$$\lim_{t \to +\infty} \left[b_t e^{-\int_0^t (r_\tau - n) d\tau} \right] = 0$$

(actualized value at time 0 of the total amount of assets in the long term $= 0$).

Transversality condition and solvency constraint

- The transversality condition implies the solvency constraint:

$$\lim_{t\to+\infty}\left[b_t e^{-\int_0^t (r_\tau - n)d\tau}\right] = 0 \Longrightarrow \lim_{t\to+\infty}\left[b_t e^{-\int_0^t (r_\tau - n)d\tau}\right] \geq 0.$$

- The solvency *constraint forbids* households from having a total debt $(-B_t > 0)$ forever increasing at a rate higher than or equal to r_t.

- The transversality *condition indicates* that it is not optimal for households to have a total amount of assets $(B_t > 0)$ forever increasing at a rate higher than or equal to r_t.

Transversality condition and intertemp. budget constraint

- Equivalently, the transversality condition implies that **the intertemporal budget constraint is binding**:

$$\lim_{t \to +\infty} \left[b_t e^{-\int_0^t (r_\tau - n) d\tau} \right] = 0 \Longrightarrow$$
$$\int_0^{+\infty} c_t e^{-\int_0^t (r_\tau - n) d\tau} dt = b_0 + \int_0^{+\infty} w_t e^{-\int_0^t (r_\tau - n) d\tau} dt.$$

- The intertemporal budget *constraint forbids* households from having an actualized value at time 0 of their future consumptions *strictly higher* than the sum of their wealth at time 0 and the actualized value at time 0 of their future wage incomes.

- The transversality *condition indicates* that it is not optimal for households to have an actualized value at time 0 of their future consumptions *strictly lower* than this sum.

Case of a constant elasticity of intertemporal substitution

- Functional form: $\begin{cases} u(c_t) = \frac{c_t^{1-\theta}-1}{1-\theta} \text{ for } \theta \in \mathbb{R}_+ \setminus \{0,1\}, \\ u(c_t) = \ln(c_t) \text{ for } \theta = 1. \end{cases}$

- The elasticity of intertemporal substitution is then constant: $\sigma(c_t) = \frac{1}{\theta}$ (we refer to this case as the CRRA case for "**Constant Relative Risk Aversion**").

- We can then rewrite the Euler equation as $\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}$ and integrate it to get

$$c_t = c_0 e^{\int_0^t \left(\frac{r_\tau - \rho}{\theta}\right) d\tau}.$$

- Replacing c_t with this expression in the binding intertemporal budget constraint, we then get

$$c_0 = \frac{b_0 + \int_0^{+\infty} w_t e^{-\int_0^t (r_\tau - n) d\tau} dt}{\int_0^{+\infty} e^{\int_0^t \left[\left(\frac{r_\tau - \rho}{\theta}\right) - (r_\tau - n)\right] d\tau} dt}.$$

Case of exponential preferences

- Functional form:
$$u(c_t) = -\alpha e^{-\frac{1}{\alpha} c_t},$$
where $\alpha > 0$.

- Part 2 of the tutorials studies this case of **exponential** preferences.

- The Euler equation can then be rewritten as $\frac{\dot{c}_t}{c_t} = \frac{\alpha}{c_t}(r_t - \rho)$, and Part 2 of the tutorials shows that one cannot then get a positive and constant growth rate of per-capita consumption in the long term.

Usage cost of capital and real interest rate

- We assume that capital depreciates at rate δ, as in Chapter 1.
- At time t, a household can in particular
  - rent from t to $t + dt$ one unit of good as capital to firms,
  - lend from t to $t + dt$ this unit of good to other households.
- At time $t + dt$, the first option brings them $z_t dt - \delta dt$ units of good, the second one $r_t dt$ units of good.
- In equilibrium, the household must be indifferent between these two options, so

$$r_t = z_t - \delta$$

(if $r_t > z_t - \delta$, then all households would want to lend and none would want to borrow; if $r_t < z_t - \delta$, then all households would want to borrow and none would want to lend; in both cases, the loans market would not be cleared).

Clearing of other markets

- In equilibrium, the representative household neither lends nor borrows, so all their assets are capital ownership titles:

$$B_t = K_t.$$

- In the labor market, demand is equal to supply:

$$N_t = L_t.$$

- The goods-market-clearing condition

$$\underbrace{Y_t}_{\text{output}} = \underbrace{C_t}_{\text{consumption}} + \underbrace{I_t}_{\text{investment}}$$

can be rewritten, using $\dot{K}_t = I_t - \delta K_t$, as

$$\underbrace{\dot{K}_t}_{\substack{\text{variation in the} \\ \text{capital stock}}} = \underbrace{Y_t - C_t}_{\text{savings}} - \underbrace{\delta K_t}_{\text{depreciation}}.$$

Equilibrium determination

Equilibrium conditions on κ_t and γ_t I

- Denoting $f(x) \equiv F(x,1)$ for all $x > 0$ and differentiating $F(K_{i,t}, A_t N_{i,t}) = A_t N_{i,t} f(\frac{K_{i,t}}{A_t N_{i,t}})$ with respect to $K_{i,t}$, we get

$$F_1(K_{i,t}, A_t N_{i,t}) = f'\left(\frac{K_{i,t}}{A_t N_{i,t}}\right).$$

- We deduce, using the first-order condition $F_1(K_{i,t}, A_t N_{i,t}) = z_t$, that $\frac{K_{i,t}}{N_{i,t}}$ does not depend on i and is therefore equal to $\frac{K_t}{N_t}$.

- As a consequence,

$$\begin{aligned} Y_t &\equiv \textstyle\sum_{i=1}^{I} Y_{i,t} = \sum_{i=1}^{I} F(K_{i,t}, A_t N_{i,t}) = \sum_{i=1}^{I} N_{i,t} F\left(\frac{K_{i,t}}{N_{i,t}}, A_t\right) \\ &= \textstyle\sum_{i=1}^{I} N_{i,t} F\left(\frac{K_t}{N_t}, A_t\right) = N_t F\left(\frac{K_t}{N_t}, A_t\right) = F(K_t, A_t N_t). \end{aligned}$$

Equilibrium conditions on κ_t and γ_t II

- Differentiating $F(K_{i,t}, A_t N_{i,t}) = A_t N_{i,t} f(\frac{K_{i,t}}{A_t N_{i,t}})$ with respect to $K_{i,t}$ and $N_{i,t}$, we get

$$F_1(K_{i,t}, A_t N_{i,t}) = f'\left(\frac{K_{i,t}}{A_t N_{i,t}}\right),$$

$$A_t F_2(K_{i,t}, A_t N_{i,t}) = A_t\left[f\left(\frac{K_{i,t}}{A_t N_{i,t}}\right) - \frac{K_{i,t}}{A_t N_{i,t}} f'\left(\frac{K_{i,t}}{A_t N_{i,t}}\right)\right].$$

- Using $\frac{K_{i,t}}{N_{i,t}} = \frac{K_t}{N_t}$, $N_t = L_t$, $\kappa_t \equiv \frac{K_t}{A_t L_t}$ and $r_t = z_t - \delta$, we can the rewrite the first-order conditions of firms' optimization problem as
  - $r_t = f'(\kappa_t) - \delta$ ("real interest rate = marginal productivity of capital − capital depreciation rate"),
  - $w_t = A_t\left[f(\kappa_t) - \kappa_t f'(\kappa_t)\right]$.

Equilibrium conditions on κ_t and γ_t III

- Using the last conditions, we can rewrite households' instantaneous budget constraint as

$$\dot{b}_t = \left[f'(\kappa_t) - \delta - n\right] b_t + A_t \left[f(\kappa_t) - \kappa_t f'(\kappa_t)\right] - c_t.$$

- Using $B_t = K_t$, we get $b_t = A_t \kappa_t$ and hence

$$\dot{\kappa}_t = f(\kappa_t) - \gamma_t - (n + g + \delta)\,\kappa_t$$

where $\gamma_t \equiv \frac{c_t}{A_t} = \frac{C_t}{A_t L_t}$ is consumption per effective-labor unit.

- This differential equation can be interpreted as "variation in the capital stock = savings − dilution − dépréciation" (per effective-labor unit) and is nothing else than the goods-market-clearing condition.

Equilibrium conditions on κ_t and γ_t IV

- The result that the goods-market-clearing condition can be derived from the other equilibrium conditions is a consequence of Walras' law.
- **Walras' law**: over all the markets, the sum of net demands weighted by prices is equal to zero.
- **Corollary of Walras' law**: in an economy with N markets, if $N-1$ markets clear, then the N^{th} market clears too.
- Walras' law thus implies that one market-clearing condition is redundant – for instance here the goods-market-clearing condition.
- **Léon Walras**: French economist, born in Évreux in 1834, deceased in Clarens in 1910.

Equilibrium conditions on κ_t and γ_t V

- Using $b_t = \kappa_t A_0 e^{gt}$, $A_0 > 0$ and $r_t = f'(\kappa_t) - \delta$, we can rewrite the transversality condition as

$$\lim_{t \to +\infty} \left\{ \kappa_t e^{-\int_0^t [f'(\kappa_\tau) - (n+g+\delta)] d\tau} \right\} = 0.$$

- From now on, we consider a constant elasticity of intertemporal substitution, equal to $\frac{1}{\theta}$.

- Using $r_t = f'(\kappa_t) - \delta$ and $\gamma_t \equiv \frac{c_t}{A_t}$, we can then rewrite the Euler equation as

$$\frac{\dot{\gamma}_t}{\gamma_t} = \frac{1}{\theta} \left[f'(\kappa_t) - \delta - \rho - \theta g \right].$$

Equilibrium conditions on κ_t and γ_t VI *

- $(\kappa_t)_{t\geq 0}$ and $(\gamma_t)_{t\geq 0}$ are therefore determined by two differential equations, one initial condition and one terminal condition:

$$\dot{\kappa}_t = f(\kappa_t) - \gamma_t - (n+g+\delta)\kappa_t,$$

$$\frac{\dot{\gamma}_t}{\gamma_t} = \frac{1}{\theta}\left[f'(\kappa_t) - \delta - \rho - \theta g\right],$$

$$\kappa_0 = \frac{K_0}{A_0 L_0},$$

$$\lim_{t\to+\infty}\left\{\kappa_t e^{-\int_0^t [f'(\kappa_\tau)-(n+g+\delta)]d\tau}\right\} = 0.$$

- The other endogenous variables are residually determined, from $(\kappa_t)_{t\geq 0}$ and $(\gamma_t)_{t\geq 0}$, using the other equilibrium conditions.

- We get the equilibrium conditions of the Solow-Swan model when the second and fourth conditions are replaced by $\gamma_t = (1-s)f(\kappa_t)$.

Steady state I

- **Steady state** $\equiv$ situation in which κ_0 is such that, in equilibrium, all quantities are non-zero and grow at constant rates.

- The differential equation in $\dot{\gamma}_t$ implies that κ_t is constant at the steady state.

- The differential equation in $\dot{\kappa}_t$ then implies that γ_t is constant at the steady state.

- Therefore, **at the steady state,**
  - κ_t **and** γ_t **are constant,**
  - $k_t \equiv \frac{K_t}{L_t} = A_t \kappa_t$ **and** $c_t \equiv \frac{C_t}{L_t} = A_t \gamma_t$ **grow at rate** g**,**
  - $y_t \equiv \frac{Y_t}{L_t} = A_t f(\kappa_t)$ **grows at rate** g**,**
  - **the saving rate** $\frac{Y_t - C_t}{Y_t} = 1 - \frac{\gamma_t}{f(\kappa_t)}$ **is constant,**

  as in the Solow-Swan model.

Steady state II

- Replacing $\dot{\kappa}_t$ with 0 in the differential equation in $\dot{\kappa}_t$, we get

$$\gamma_t = f(\kappa_t) - (n + g + \delta)\,\kappa_t,$$

which corresponds to a **bell-shaped curve** in the plane (κ_t, γ_t).

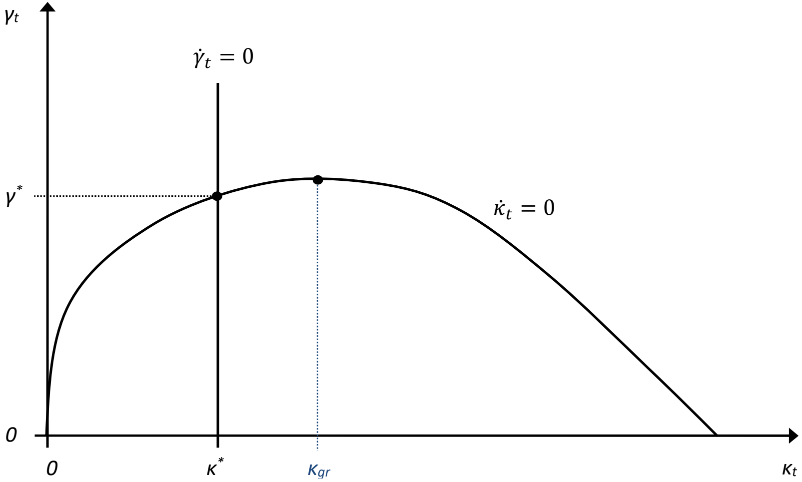
- Replacing $\dot{\gamma}_t$ with 0 in the differential equation in $\dot{\gamma}_t$, we get

$$f'(\kappa_t) = \delta + \rho + \theta g,$$

which corresponds to a **vertical straight line** in the plane (κ_t, γ_t).

- The **intersection point** of this curve and this straight line corresponds to the steady-state value of (κ_t, γ_t), denoted by (κ^*, γ^*).

Steady state III



Steady state IV

- The value κ_{gr} of κ_t maximizing the bell-shaped curve is defined by

$$f'(\kappa_{gr}) = n + g + \delta$$

(**golden rule of capital accumulation** of the Solow-Swan model).

- The conditions $\kappa^* = \kappa_0 > 0$ and

$$\lim_{t \to +\infty} \left\{ \kappa^* e^{-\int_0^t [f'(\kappa^*) - (n+g+\delta)] d\tau} \right\} = 0$$

imply $f'(\kappa^*) > n + g + \delta = f'(\kappa_{gr})$ and hence $\kappa^* < \kappa_{gr}$, as apparent in the previous figure.

Steady state V

- We do not have $\kappa^* > \kappa_{gr}$, so we do not have **dynamic inefficiency** (due to capital over-accumulation), because of households' optimizing behavior.

- We have $\kappa^* < \kappa_{gr}$ and not $\kappa^* = \kappa_{gr}$ because households are **sufficiently impatient** for $\rho - n > (1 - \theta) g$ (necessary condition for households' intertemporal utility to take a finite value at the steady state).

- Because of households' impatience, when $\kappa_t > \kappa^*$, a decrease in savings raises the short-term component of intertemporal utility more than it reduces its long-term component.

- The equation $f'(\kappa^*) = \delta + \rho + \theta g$ is called the "**modified golden rule**".

Steady state VI

- Differentiating $f'(\kappa^*) = \delta + \rho + \theta g$ and $\gamma^* = f(\kappa^*) - (n + g + \delta)\kappa^*$ with respect to ρ, θ, g, δ or n, and using $f'(\kappa^*) > n + g + \delta$, we get that
  - κ^* **is strictly decreasing in** ρ**,** θ**,** g**,** δ**, and constant in** n**,**
  - γ^* **is strictly decreasing in** ρ**,** θ**,** g**,** δ**,** n**.**

- These analytical results can be illustrated graphically in the previous figure:
  - a rise in n moves the bell-shaped curve downwards,
  - a rise in ρ or θ moves the straight line leftwards,
  - a rise in g or δ does both.

- Interpretation of the effect of a rise in θ on κ^*:

$$\theta \uparrow \Rightarrow \text{elasticity of intertemporal substitution} \downarrow \Rightarrow r_t \uparrow$$
$$\text{to make households accept } \frac{\dot{c}_t}{c_t} = g \Rightarrow f'(\kappa_t) \uparrow \Rightarrow \kappa_t \downarrow.$$

Steady state VII

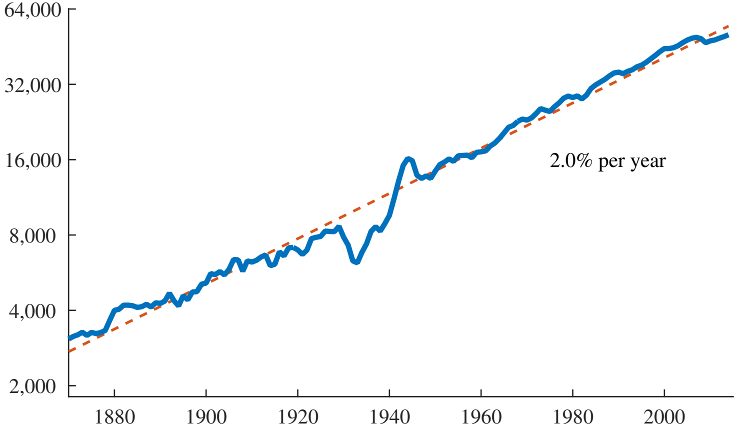
- At the steady state, the Cass-Koopmans-Ramsey model therefore accounts for the first five "**stylised facts**" (i.e. empirical regularities) identified by **Kaldor (1961)**:

1. per-capita output grows: $\frac{\dot{y}_t}{y_t} = g \geq 0$,
2. the per-capita capital stock grows: $\frac{\dot{k}_t}{k_t} = g \geq 0$,
3. the rate of return of capital is constant: $r_t = f'(\kappa^*) - \delta$,
4. the ratio capital / output is constant: $\frac{K_t}{Y_t} = \frac{\kappa^*}{f(\kappa^*)}$,
5. the labor and capital shares of income are constant: $\frac{w_t L_t}{Y_t} = \frac{f(\kappa^*) - \kappa^* f'(\kappa^*)}{f(\kappa^*)}$ and $\frac{z_t K_t}{Y_t} = \frac{\kappa^* f'(\kappa^*)}{f(\kappa^*)}$

(the sum of the factors' shares of income is equal to one because under perfect competition, all the firms' benefits are used to pay for the factors, as we saw on page 17).

Kaldor's stylised fact No. 1

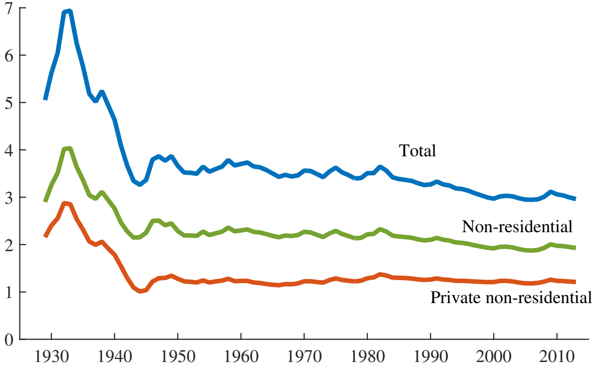
Per-capita GDP in the US, 1870-2014
(logarithmic scale, 2009 dollars)



Source: Jones (2015).

Kaldor's stylised fact No. 4

Ratio of physical capital to GDP in the US, 1929-2014
(depending on the measure of physical capital considered)

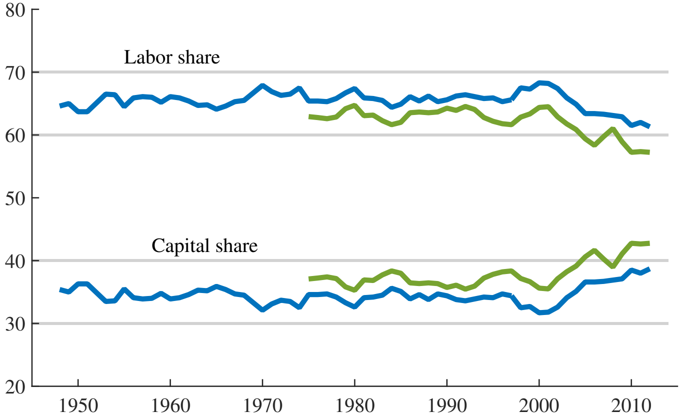


Source: Jones (2015).

Remark: Kaldor's stylised facts No. 1 and 4 imply Kaldor's stylised fact No. 2.

Kaldor's stylised fact No. 5

Capital and labor shares of factor payments in the US
(in blue/green: with/without self-employed, 1948-2014/1975-2014, in %)



Source: Jones (2015).

Steady state VIII

- At the steady state, the Solow-Swan model accounts for the 1$^{\text{st}}$, 2$^{\text{nd}}$ and 4$^{\text{th}}$ stylised facts of Kaldor (1961), but not for the 3$^{\text{rd}}$ and 5$^{\text{th}}$ ones (because it does not consider the prices w_t, z_t and r_t).

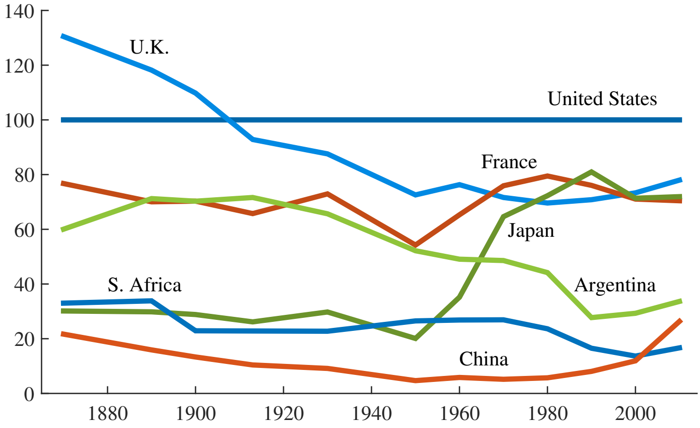
- At the steady state, neither the Solow-Swan model nor the Cass-Koopmans-Ramsey model easily account for the 6$^{\text{th}}$ stylised fact of Kaldor (1961):

  6. the growth rate of per-capita output varies across countries,

  because they imply that the growth rate $\frac{\dot{y}_t}{y_t}$ is equal to the rate of technological progress g which is probably not permanently different across countries (due to the possibility of knowledge diffusion across countries).

Kaldor's stylised fact No. 6 I

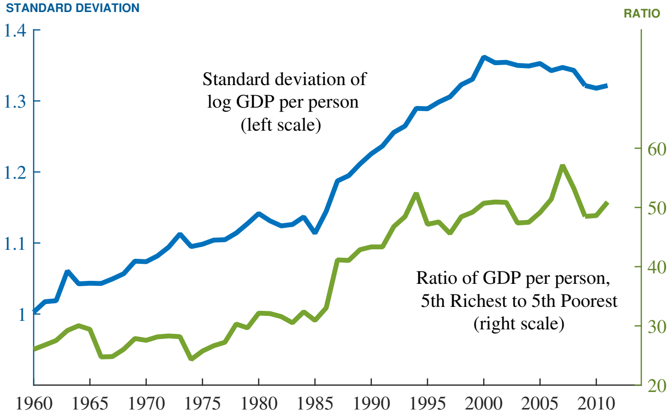
Per-capital GDP in various countries, 1870-2014 (US = 100)



Source: Jones (2015).

Kaldor's stylised fact No. 6 II

Dispersion of per-capita GDP across one hundred countries, 1960-2011



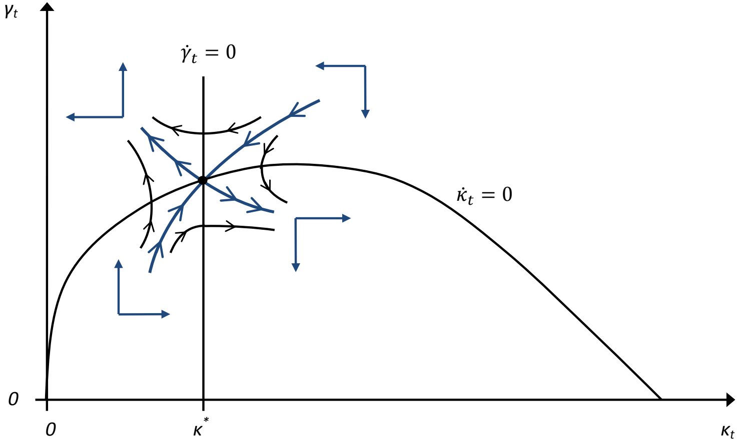
Source: Jones (2015).

Convergence to steady state I

- The differential equation in $\dot{\kappa}_t$ implies that
  - **below** the bell-shaped curve, κ_t **increases** over time,
  - **above** the bell-shaped curve, κ_t **decreases** over time.
- The differential equation in $\dot{\gamma}_t$ implies that
  - **to the left** of the straight line, γ_t **increases** over time,
  - **to the right** of the straight line, γ_t **decreases** over time.
- The system of differential equations has therefore a stable arm and an unstable arm in the plane (κ_t, γ_t).

Convergence to steady state II

Phase diagram: shape of the paths satisfying the two differential equations (but not necessarily the initial and terminal conditions)



Convergence to steady state III

- In the neighborhood of the steady state, the stable (resp. unstable) arm corresponds to the eigenvector associated to the negative (resp. positive) eigenvalue of the matrix M of the log-linearized system of differential equations

$$\begin{bmatrix} \dot{\ln\left(\frac{\kappa_t}{\kappa^*}\right)} \\ \ln\left(\frac{\gamma_t}{\gamma^*}\right) \end{bmatrix} = \underset{(2\times 2)}{M} \begin{bmatrix} \ln\left(\frac{\kappa_t}{\kappa^*}\right) \\ \ln\left(\frac{\gamma_t}{\gamma^*}\right) \end{bmatrix}.$$

- The stable arm is the unique path, called the "**saddle path**", along which (κ_t, γ_t) can converge to (κ^*, γ^*).

Convergence to steady state IV

- For a given $\kappa_0 \in (0, \kappa^*)$, there are three alternative cases for γ_0:

1. γ_0 is such that (κ_0, γ_0) is **on** the saddle path:
   - (κ_t, γ_t) converges to (κ^*, γ^*);
   - the four equilibrium conditions are satisfied.

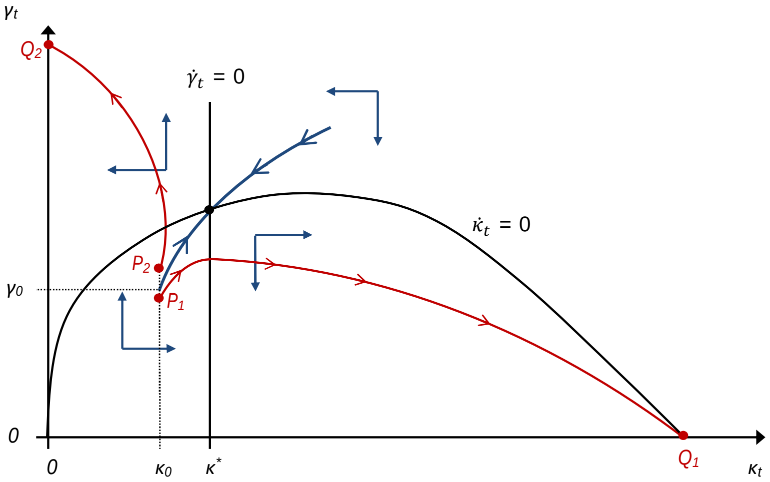
2. γ_0 is such that (κ_0, γ_0) is **below** the saddle path (Point P_1):
   - (κ_t, γ_t) crosses the vertical straight line and then converges to Point Q_1 (the reason being that, in the right lower quadrant, (κ_t, γ_t) cannot cross the bell-shaped curve because the slope would then be vertical, nor can it cross the x axis because consumption cannot be negative);
   - in the neighborhood of Q_1, $\kappa_t > \kappa_{gr}$, so $f'(\kappa_t) < f'(\kappa_{gr}) = n + g + \delta$, so the transversality condition is not satisfied:

$$\lim_{t \to +\infty} \left\{ \kappa_t e^{-\int_0^t [f'(\kappa_\tau) - (n+g+\delta)] d\tau} \right\} > 0$$

     (households, who asymptotically save everything and consume nothing, could raise their utility level by saving less).

Convergence to steady state V

Phase diagram: paths for a given $\kappa_0 \in (0, \kappa^*)$



Convergence to steady state VI

3. γ_0 is such that (κ_0, γ_0) is **above** the saddle path (Point P_2):
   - (κ_t, γ_t) crosses the bell-shaped curve and then reaches Point Q_2 in finite time because $\ddot{\kappa}_t < 0$ in the left upper quadrant;
   - proof that $\ddot{\kappa}_t < 0$ in this quadrant: differentiate $\dot{\kappa}_t = f(\kappa_t) - \gamma_t - (n+g+\delta)\kappa_t$ and get $\ddot{\kappa}_t = [f'(\kappa_t) - (n+g+\delta)]\dot{\kappa}_t - \dot{\gamma}_t$ with, in this quadrant, $\dot{\kappa}_t < 0$, $\dot{\gamma}_t > 0$ and $f'(\kappa_t) > f'(\kappa_{gr}) = n+g+\delta$;
   - at the time when Point Q_2 is reached, γ_t becomes instantaneously zero (because $Y_t = 0$), so the differential equation in $\dot{\gamma}_t$ is not satisfied (since $\dot{\gamma}_t$ does not exist at that time): households consume more and more by dissaving more and more until there is nothing left to consume, which is not optimal.

- So, **the unique equilibrium path** (that is to say the unique path satisfying the four equilibrium conditions) **is the saddle path**.

- The reasoning is similar, and the conclusion is identical, when $\kappa_0 > \kappa^*$.

Convergence to steady state VII

- The model therefore predicts a **conditional convergence** of $\ln(y_t)$ across countries, like the Solow-Swan model.

- More precisely, it predicts the long-term convergence of $\ln(y_t)$ across countries that have different y_0s but the same
  - technology parameters A_0, g, $f(.)$,
  - parameters governing the dynamics of capital and labor n, δ,
  - preference parameters ρ, θ.

- We admit that, for $\kappa_0 < \kappa^*$,
  - the saving rate can increase or decrease over time,
  - the growth rate always decreases over time,

  so an economy grows all the more rapidly as it is far away from its steady-state path, like in the Solow-Swan model.

- Part 2 of the tutorials studies the speed of convergence in the neighborhood of the steady state, as well as the path followed by the economy following a permanent fall in ρ.

Equilibrium optimality

Social optimality of the competitive equilibrium I

- **Market equilibrium (or decentralized allocation)** $\equiv$ equilibrium obtained when
  - agents interact with each other on markets,
  - each agent solves their own optimization problem,
  - markets clear.

- **Competitive equilibrium** $\equiv$ market equilibrium when all markets are perfectly competitive (like in the Cass-Koopmans-Ramsey model).

- So, a competitive equilibrium is an equilibrium obtained when
  - agents interact with each other on perfectly competitive markets,
  - each agent solves their own optimization problem, choosing quantities and considering prices as given,
  - prices are such that markets clear.

Social optimality of the competitive equilibrium II

- **Representative-agent model** $\equiv$ model in which only one type of agent, called representative agent, has a utility function (like the Cass-Koopmans-Ramsey model, in which only households have a utility function).

- In representative-agent models, the market equilibrium is said to be **socially optimal** if and only if it coincides with the allocation chosen by the **benevolent, omniscient and omnipotent planner** (denoted by $\mathcal{BOOP}$), also called "centralized allocation".

Social optimality of the competitive equilibrium III

- The $\mathcal{BOOP}$, who is a fictive character, chooses all quantities subject to
  - the non-negativity constraint,
  - the technology constraint,
  - the resource constraint,

  in order to maximize the utility of the representative agent.

- Since the market equilibrium satisfies these three constraints, the value taken by the representative agent's utility function with the $\mathcal{BOOP}$ is necessarily higher than or equal to the value that it takes in the market equilibrium.

Social optimality of the competitive equilibrium IV

- Optimization problem of the $\mathcal{BOOP}$: for a given $\kappa_0 > 0$,

$$\max_{(c_t)_{t\geq 0},(\kappa_t)_{t>0}} \int_0^{+\infty} e^{-(\rho-n)t} u(c_t) dt$$

subject to the constraints

1. $\forall t \geq 0$, $c_t \geq 0$ (non-negativity of consumption),
2. $\forall t > 0$, $\kappa_t \geq 0$ (non-negativity of capital),
3. $\forall t \geq 0$, $\dot{\kappa}_t = f(\kappa_t) - \frac{c_t}{A_0 e^{gt}} - (n+g+\delta)\kappa_t$ (technology and resource constraint).

- We solve this dynamic-optimization problem by applying the optimal-control theory, as previously in this chapter.

Social optimality of the competitive equilibrium V

- **Hamiltonian** associated with the optimization problem of the $\mathcal{BOOP}$:

$$H^p(c_t, \kappa_t, \lambda_t^p, t) \equiv e^{-(\rho-n)t} u(c_t) + \lambda_t^p \left[f(\kappa_t) - \frac{c_t}{A_0 e^{gt}} - (n+g+\delta)\kappa_t \right]$$

where λ_t^p represents the value, measured in utility units at time 0, of an increase of one unit of good in the resources at time t.

- We then get the following optimality conditions:
  - $\lambda_t^p = A_0 e^{(n+g-\rho)t} u'(c_t)$ (1^{st}-order cond. on the control var.),
  - $\dot{\lambda}_t^p = [n+g+\delta - f'(\kappa_t)] \lambda_t^p$ (costate equation),
  - $\dot{\kappa}_t = f(\kappa_t) - \frac{c_t}{A_0 e^{gt}} - (n+g+\delta)\kappa_t$ (resource constraint),
  - $\lim_{t \to +\infty} \kappa_t \lambda_t^p = 0$ (transversality condition).

Social optimality of the competitive equilibrium VI

- Manipulating these conditions in the same way as previously, we get
  - $\dot{\kappa}_t = f(\kappa_t) - \gamma_t - (n + g + \delta)\kappa_t$ (differential equation in $\dot{\kappa}_t$),
  - $\frac{\dot{\gamma}_t}{\gamma_t} = \frac{1}{\theta}\left[f'(\kappa_t) - \delta - \rho - \theta g\right]$ (differential equation in $\dot{\gamma}_t$),
  - $\lim_{t \to +\infty} \left\{ \kappa_t e^{-\int_0^t [f'(\kappa_\tau) - (n+g+\delta)] d\tau} \right\} = 0$ (transversality condition).
- These three conditions and the condition $\kappa_0 = \frac{K_0}{A_0 L_0}$ are identical to the four competitive-equilibrium conditions on $(\kappa_t)_{t \geq 0}$ and $(\gamma_t)_{t \geq 0}$: **the competitive equilibrium is therefore socially optimal**.
- As a consequence, there is no role to play for economic policy in this model: the optimal economic policy is **laisser-faire**.

Social optimality of the competitive equilibrium VII

- The social optimality of the competitive equilibrium in this model is a consequence of the **first welfare theorem**.
- According to this theorem, if
  1. there is no externality,
  2. markets are perfectly competitive,
  3. markets are "complete" ($\equiv$ with non-zero demand and supply),
  4. the number of agent types is finite,

  then **the competitive equilibrium is a Pareto optimum**.
- **Pareto optimum** $\equiv$ situation in which one cannot increase the utility of one agent without decreasing the utility of another agent ("rob Peter to pay Paul").
- **Vilfredo Pareto**: Italian sociologist and economist, born in 1848 in Paris, deceased in 1923 in Céligny.

Social optimality of the competitive equilibrium VIII

- This theorem formalizes Smith's (1776) concept of "invisible hand", according to which individual actions only guided by personal interest can contribute to the wealth and welfare of all:

"*But the annual revenue of every society is always precisely equal to the exchangeable value of the whole annual produce of its industry, or rather is precisely the same thing with that exchangeable value. As every individual, therefore, endeavours as much as he can both to employ his capital in the support of domestic industry, and so to direct that industry that its produce may be of the greatest value; every individual necessarily labours to render the annual revenue of the society as great as he can. He generally, indeed, neither intends to promote the public interest, nor knows how much he is promoting it.*

Social optimality of the competitive equilibrium IX

By preferring the support of domestic to that of foreign industry, he intends only his own security; and by directing that industry in such a manner as its produce may be of the greatest value, he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention. Nor is it always the worse for the society that it was no part of it. By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it. I have never known much good done by those who affected to trade for the public good. It is an affectation, indeed, not very common among merchants, and very few words need be employed in dissuading them from it."

- **Adam Smith**: Scottish philosopher and economist, born in 1723 in Kirkcaldy, deceased in 1790 in London.

Social optimality of the competitive equilibrium X

- In representative-agent models, like the Cass-Koopmans-Ramsey model,
  - there exists a unique symmetric Pareto optimum (symmetric in the sense that the utility function takes the same values for all agents),
  - this Pareto optimum corresponds to the allocation that maximizes the representative agent's welfare (i.e. the allocation chosen by the $\mathcal{BOOP}$).

- The four conditions under which the theorem applies are met in the Cass-Koopmans-Ramsey model (present chapter), but not in
  - the DICE model (Chapter 3, ~~Condition 1~~),
  - the model with learning by doing (Chapter 4, ~~Condition 1~~),
  - the model with product variety (Chapter 5, ~~Condition 2~~),
  - the overlapping-generations model (Chapter 7, ~~Conditions 1 and 4~~).

Environmental extensions

Taking non-renewable natural resources into account I

- The CKR model, in which the only production factors are capital and labor, does not enable one to study the consequences of the exploitation of **non-renewable natural resources** (oil, natural gas, coal, minerals...).
- For most non-renewable natural resources, the known world reserves that are exploitable at current prices correspond to a few decades of exploitation.
- Part 3 of the tutorials introduces non-renewable natural resources in the CKR model, as a third production factor, and studies the positive and normative implications.

Taking non-renewable natural resources into account II

- Part 3 of the tutorials notably shows that
  - in the competitive equilibrium, the growth rate of the price of non-renewable natural resources is equal to the interest rate ("**Hotelling's rule**", after Hotelling, 1931),
  - the planner chooses a per-capita-consumption path that decreases to zero over time, which harms future generations,
  - in order to implement the allocation chosen by a planner under an intergenerational-equity constraint (requiring a constant-over-time per-capita consumption), one has to re-invest the rent extracted from the exploitation of non-renewable resources into physical capital ("**Hartwick's rule**", after Hartwick, 1977).

Growth and climate change

- The CKR model does not take into account the consequences of economic activity for the climate nor, vice-versa, the consequences of climate change for the economy.

- Nordhaus (1992, 1994) has extended the CKR model to take these consequences into account, giving rise to the **DICE model** ($\equiv$ Dynamic Integrated Climate-Economy model), which is a model of the world economy and the world climate.

- The DICE model does not have the same positive implications, nor the same normative implications, as the CKR model.

- Chapter 3 presents the DICE model and studies its normative implications.

Conclusion

Main predictions of the model

- As in the Solow-Swan model, in the long term,
  - growth stems uniquely from technological progress,
  - the effect of capital accumulation on growth disappears because of the decreasing marginal productivity of capital,
  - there is conditional convergence of per-capita output levels (in logarithm) across countries.
- In the long term, the first five stylised facts of Kaldor (1961) are obtained.
- The competitive equilibrium is socially optimal; in particular, there cannot be dynamic inefficiency due to capital over-accumulation.

One limitation of the model

- **The rate of technological progress g is exogenous.** If it were endogenous,
  - could some policies affect it?
  - what role should they play?

$\hookrightarrow$ Chapters 4 and 5 ("endogenous-growth theories") endogenize the rate of technological progress.

Appendix

Interpretation of optimality conditions (households) I

- As a reminder, the Hamiltonian associated to households' optim. problem is

$$H(c_t, b_t, \lambda_t, t) \equiv e^{-(\rho-n)t} u(c_t) + \lambda_t \left[(r_t - n) b_t + w_t - c_t\right]$$

and the corresponding optimality conditions are:

1. $\forall t \geq 0$, $\frac{\partial H}{\partial c_t} = 0$ (first-order condition on the control variable),
2. $\forall t \geq 0$, $\frac{\partial H}{\partial b_t} = -\dot{\lambda}_t$ (costate equation),
3. $\forall t \geq 0$, $\dot{b}_t = (r_t - n) b_t + w_t - c_t$ (instantaneous budget constraint),
4. $\lim_{t \to +\infty} b_t \lambda_t = 0$ (transversality condition),
5. $\lim_{t \to +\infty} \left[b_t e^{-\int_0^t (r_\tau - n) d\tau}\right] \geq 0$ (solvency constraint),
6. $\forall t \geq 0$, $c_t \geq 0$ and $\lambda_t \geq 0$ (non-negativity constraints).

Interpretation of optimality conditions (households) II

- Conditions 3, 5 and "6a" are the constraints of the optimization problem.
- Condition "6b" indicates that λ_t, which measures the utility at time 0 of one unit of savings at time t, cannot be negative.
- Condition 4 implies that $\lim_{t\rightarrow+\infty} b_t\lambda_t$, which measures the utility at time 0 of the stock of savings at time $t \rightarrow +\infty$, cannot be positive at the optimum (otherwise households could increase their intertemporal utility by saving less and consume more in the long term).
- Condition 1 can be rewritten as

$$\lambda_t = e^{-(\rho-n)t} u'(c_t)$$

and can be interpreted as follows: at the optimum, the utility at time 0 of one unit of savings at time t must be equal to the utility at time 0 of one unit of consumption at time t, otherwise households could increase their intertemporal utility by choosing another sharing between consumption and savings.

Interpretation of optimality conditions (households) III

- Condition 2 can be rewritten as $\dot{\lambda}_t = -(r_t - n)\lambda_t$.
- Assume that households marginally deviate from the optimum as follows:
  - at time t, they consume one less unit and save one more unit,
  - after time t, they keep this additional unit of savings and consume the financial incomes that it generates.
- In terms of utility at time 0, this marginal deviation
  - has a cost of $e^{-(\rho-n)t}u'(c_t) = \lambda_t$,
  - brings a benefit of $\int_t^{+\infty} e^{-(\rho-n)\tau}u'(c_\tau)(r_\tau - n)\,d\tau = \int_t^{+\infty}\lambda_\tau(r_\tau - n)\,d\tau$,

  where the equalities are obtained by using Condition 1.
- Since it is a marginal deviation from the optimum, this cost and this benefit must be equal to each other: $\lambda_t = \int_t^{+\infty}\lambda_\tau(r_\tau - n)\,d\tau$.
- Differentiating with respect to t, we get Condition 2: $\dot{\lambda}_t = -(r_t - n)\lambda_t$.